The General Equilibrium of Elastic Layered Systems (GELS)
An Open-Source Implementation in Python

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1 Abstract

Hankel Transforms of the biharmonic operator on a stress function are used to solve for the stresses and displacements in an elastic body under an axisymmetric load using cylindrical coordinates. The stress function of a fourth order partial differential equation reduces to a transformed second order ordinary differential equation with repeated roots. Four constant coefficients are determined from the boundary conditions. Solving the multiple elastic layer problem requires finding the four constant coefficients for each layer at each value of the Hankel transform variable. This implementation follows the solution as documented in Crawford, Hopkins and Smith, Theoretical Relationships between Moduli for Soil Layers beneath Concrete Pavements.

The one layer solution requires finding only two constant coefficients from the boundary conditions and can be solved analytically for a normal, uniform, circular surface load. Substituting the two constant coefficients into the Inverse Hankel Transform to get the stresses and displacements, solutions in integrals of Bessel functions of the first kind, order zero and one are found. The Laplace Transforms of those Bessel functions matches the solutions of Boussinesq and Egorov, as documented in Harr, Foundations of Theoretical Soil Mechanics.

This implementation will be used to plot the stresses and displacements of an elastic layered system with layer thickness as a variable and the subgrade modulus as a variable under a multiple wheeled vehicle.

The origin of the implementation is from the University of Illinois, Urbana-Champaign, Multiple Wheel Elastic Layer Program (MWELP), from the early 1970s.

2 Introduction

Purpose of a pavement is to provide a functional surface for the safe operation of a vehicle.
The operator of the vehicle does not care what material the pavement structure consists of, but is sensitive to the vehicle rattling/vibrating at the speed of travel, and is aware of the rough country road compared to a smooth highway.

A scientific solution for designing a suitable pavement structure, would require an engineering model to determine the deflections and stresses for a vehicle load on an elastic layered system as characterized by the layers' engineering properties, modulus of elasticity, poisson's ratio and geometry (thickness).
From the engineering model and experimental data, the elastic deflections and stresses can be used to predict the permanent deformation and the fatigue life of the pavement layers.
Vehicle Speed/Repetitions/Pavement Roughness, elastic deflection to permanent deformation
Vehicle Weight/Repetitions/Pavement Maintenance/Repair, elastic stress to fatigue life

3 Why use elasticity theory as a mathematical model for layered systems on a soil mass

Theoretical Soil Mechanics
Karl Terzaghi
Chapter XVII
Theory of Semi-Infinite Elastic Solids
132. Elastic and plastic equilibrium.
If the factor of safety of a mass of soil with respect to failure by plastic flow (see Section B) exceeds a value of about 3 the state of stress in the soil is likely to be more or less similar to the state of stress computed on the assumption that the soil is perfectly elastic. Hence the state of stress in a mass of soil under the influence of moderate stresses can be estimated by the means of the theory of elasticity. The importance of the error associated with the results of the computations depends chiefly on the extent to which the real stress-strain relations depart from Hooke's law. This departure increases rapidly as the state of plastic equilibrium is approached. If the departure can be expected to be an unimportant one can use the theory of elasticity as described in this chapter. If it is likely to be important one has to use the theory of plasticity in accordance with the procedures described in Chapters V to XI.

Stress: force per unit of area
Strain: change of length per unit of length in a given direction
Isotropy: identical elastic properties throughout the solid and in every direction through any point in it
Homogeneity: identical elastic properties at every point of the solid in identical directions
Hooke's law: ratio between a linear stress and the corresponding linear strain is a constant, called modulus of elasticity or Young's modulus

Theoretical Relationships between Moduli for Soil Layers beneath Concrete Pavements
John E. Crawford
Jerome S. Hopkins
James Smith
FAA-RD-75-140
The elastic layer idealization considers a semi-infinite body composed of N horizontal layers of homogeneous material. A uniform pressure P is applied over a circular area of radius "a" to the top surface of the top layer. Each layer is defined using Young's modulus E, Poisson's ration u, and layer thickness, h.
Appendix A
Mathematical Description for the Multi-Layer Elastic Problem
The multi-layer elastic problem consists of N layers of homogeneous linear elastic material of infinite lateral extent. The layers are numbered from top to bottom. Each layer (n) has a Young's modulus (En), a Poisson's ration (un), and a thickness (hn: except for the Nth layer which has an infinite depth). A uniform pressure (P) is applied over a circular are of radius (a) to the top surface of the top layer. The problem is to find the downward displacement [w(0,0)] at T and the lateral stress [Srr(0,h1)] at B, where T is the origin of a cylindrical coordinate system (R, Z). The Z coordinate is positive downward, and for this problem the spatial coordinate R will always be zero. The layers are taken to be bonded at their interfaces.

Stress and Displacement Characteristics of a Two-Layer Rigid Base Soil System: Influence Diagrams and Practical Applications
Donald M. Burmister

Force at a Point in the Interior of a Semi-Infinite Solid
R. D. Mindlin

Design of Functional Pavements
Nai C. Yang
Chapter Seven
Mathematical Models for Pavement System
A. Equilibrium of Pavement Systems
Mathematical models are the tools by which engineers apply scientific principles to the solution of engineering problems even without the benefit of past experience. The solution is based on the physical requirements of a structure to withstand the anticipated external loads, postulated deformations and stresses in the elements, and the mechanical behavior of materials according to the basic laws of mechanics governing motion and force. Thus, a mathematical model consists

of three submodels:
1. The equilibrium of the pavement system under the influence of external loads.
2. For a given supported condition, an evaluation of the deformations and stresses in the pavement elements.
3. A characterization of the fundamental properties of pavement materials and their effect on the equilibrium and stability of the pavement structure.
7.1. General Equilibrium Equations
In studying the equilibrium of an elastic body, it is assumed that the body will not move as a rigid body, so that no displacement of particles of the body is possible without a deformation of the body.
7.2. Force on Boundary of a Semi-Infinite Body
The solution of Eq. (11) can be obtained by assuming that the stress function is a series of polynomials.
Since the solution was first given by J. Boussinesq in 1885, Eqs. (22) are known as Boussinesq theories.
7.6. Layered Systems
Since pavements normally consist of several layers of material, it is natural to consider the theory of layered systems. The Boussinesq equations are theoretically sound for one-layer systems—a semi-infinite mass having a distributed load on the boundary surface. Although actual measurements have demonstrated that the deflection of a pavement system is in good agreement with the deflection Wz computed by the Boussinesq equation (32), the hypothesis remains theoretically unthinkable (see Fig. 7.6). In recent years, considerable effort has been expended on the analysis of stresses and displacements in multiple-layered systems such as the system shown in Fig. 7.8. Most of the analyses include certain basic assumptions, which can be summarized as follows: (1) each layer is composed of materials which are isotropic, homogeneous, and weightless; (2) the systems acts as a composite system, that is, there is a continuity of stresses and/or displacements across the interfaces, depending upon the assumptions made regarding the interface conditions; and (3) most solutions assume materials which are linearly elastic.
The first solution for a generalized multiple-layered elastic system was presented by Burmister [13, 14, 15]. In this series of papers, Burmister formulated the problem of N-layered elastic systems and developed solutions for specific two- and three-layered systems. Burmister's work was limited to uniform, normal loads applied over a circular area. Schiffman [48] later extended Burmister's work for more generalized asymmetric loading conditions, including shear stresses at the surface.

Static and Dynamic Analysis of Structures
Edward L. Wilson
1. Stress-strain relationship contains the material property information that must be evaluated by laboratory or field experiments. Mechanical material properties for most common materials are defined in terms of three numbers: modulus of elasticity E, Poisson's ratio, u and coefficient of thermal expansion. T. In addition, the unit weight w and the unit mass, m, are considered to be fundamental properties.

A Treatise on the Mathematical Theory of Elasticity
A. Love
188. Symmetrical Strain in a Solid of Revolution
The stress-components are now expressed in terms of a single function, ϕ which satisfies the equation (65)*
My note: (65) is the biharmonic operator on $\nabla^4\phi = 0$

Boussinesq, single layer versus GELS, multiple layers
Calculate vertical stress and deflection at various depths
GELS, calculate horizontal stress and deflection for different layers with varying material properties, brittle to ductile
Boussinesq, thickness design uses rule of thumb limiting vertical stress in subgrade
GELS, thickness design is based on limiting horizontal stress in layered materials and limiting surface deflection to reduce permanent deformation.
Some typical modulus of elasticity for soils can be found in a paper by Obrzud and Truty 2012
If stresses from repeated loads are kept below the endurance limit, then the material has a theoretically infinite life.
Does soil have an endurance limit?
Probably not but to increase performance/life, reduce stress/deformation in soil mass.
With elastic stresses and deflections from GELS, some experimental relationships to fatigue life and permanent deformation have been proposed.
The fatigue life relationship is based on the ultimate material strength and number of repeated loads.
Sn = (1 -FATIST) * log10(N) * ULSTR where ULSTR = STRESS * E ** STRFCTR
The permanent deformation relationship is based on the rate of permanent deformation due to repeated loads.
Dn = D0 * log10(N) + D1 and the elastic to rate of permanent deformation
Wz/W0 = B0 * D0/W0 + B1 where Wz is the GELS, multiple wheel surface deflection and W0 is the Boussinesq single

wheel subgrade surface deflection.
An empirical relationship between a material's Poisson's ratio and modulus of elasticity is used as
u = 0.65 - 0.08 * log10(E)

4 GELS Formulas

$$
\begin{aligned}
\nabla^4 \phi &= 0 \\
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \\
\text{vertical stress; } \sigma_{zz} &= \frac{\partial}{\partial z}\left[(2-\mu)\nabla^2\phi - \frac{\partial^2 \phi}{\partial z^2}\right] \\
\text{shear stress; } \sigma_{rz} &= \frac{\partial}{\partial r}\left[(1-\mu)\nabla^2\phi - \frac{\partial^2 \phi}{\partial z^2}\right] \\
\text{radial stress; } \sigma_{rr} &= \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{\partial^2 \phi}{\partial r^2}\right] \\
\text{horizontal stress; } \sigma_{\theta\theta} &= \frac{\partial}{\partial z}\left[\mu\nabla^2\phi - \frac{1}{r}\frac{\partial \phi}{\partial r}\right] \\
\text{vertical displacement; } w &= \frac{1+\mu}{E}\left[2(1-\mu)\nabla^2\phi - \frac{\partial^2 \phi}{\partial z^2}\right] \\
\text{horizontal displacement; } u &= -\frac{1+\mu}{E}\left[\frac{\partial^2 \phi}{\partial r \partial z}\right] \\
\text{zero order Hankel transform; } L_0(g) = \bar{g} &= \int_0^\infty g r J_0(pr) dr \\
\text{first order Hankel transform; } L_1\left(\frac{\partial g}{\partial r}\right) &= \int_0^\infty \frac{\partial g}{\partial r} r J_1(pr) dr \\
&= -pL_0\left(g\right) \\
L_0\left(\nabla^4\phi\right) &= \left[\frac{d^2\bar{\phi}}{dz^2} - p^2\bar{\phi}\right]^2 = 0
\end{aligned}
$$

General solution for this ordinary differential equation is;

$$\bar{\phi} = [\alpha_1 + \alpha_3 z] e^{pz} + [\alpha_2 + \alpha_4 z] e^{-pz}$$

$$L_0^{-1}\left(\bar{\phi}\right) = \phi = \int_0^\infty \bar{\phi} p J_0(pr)\, dp$$

$$\frac{1}{r}\frac{\partial \phi}{\partial r} = -\int_0^\infty \bar{\phi} p^3 \frac{J_1(pr)}{pr} dp$$

$$\frac{\partial^2 \phi}{\partial r^2} = \int_0^\infty \bar{\phi} p^3 \frac{J_1(pr)}{pr} dp - \int_0^\infty \bar{\phi} p^3 J_0(pr)\, dp$$

$$\frac{\partial \phi}{\partial z} = \int_0^\infty \left([\alpha_1 p + \alpha_3(1+pz)] e^{pz} - [\alpha_2 p + \alpha_4(1-pz)] e^{-pz}\right) p J_0(pr)\, dp$$

$$\frac{\partial^2 \phi}{\partial z^2} = \int_0^\infty \left(\left[\alpha_1 p^2 + \alpha_3\left(2p + p^2 z\right)\right] e^{pz} + \left[\alpha_2 p^2 + \alpha_4\left(-2p + p^2 z\right)\right] e^{-pz}\right) p J_0(pr)\, dp$$

$$\frac{\partial^3 \phi}{\partial z^3} = \int_0^\infty \left(\left[\alpha_1 p^3 + \alpha_3\left(3p^2 + p^3 z\right)\right] e^{pz} - \left[\alpha_2 p^3 + \alpha_4\left(-3p^2 + p^3 z\right)\right] e^{-pz}\right) p J_0(pr)\, dp$$

$$\nabla^2 \phi = \int_0^\infty \left(\alpha_3 2p e^{pz} - \alpha_4 2p e^{-pz}\right) p J_0(pr)\, dp$$

$$\frac{\partial J_0(pr)}{\partial r} = -p J_1(pr)$$

Stresses and displacements in terms of the Hankel transformed stress function;

$$\sigma_{zz} = \int_0^\infty \left(\left[-\alpha_1 p^3 - \alpha_3 p^2 (pz + 2\mu - 1)\right] e^{pz} + \left[\alpha_2 p^3 + \alpha_4 p^2 (pz - 2\mu + 1)\right] e^{-pz}\right) p J_0(pr)\, dp$$

$$\sigma_{rz} = \int_0^\infty \left(\left[\alpha_1 p^3 + \alpha_3 p^2 (pz + 2\mu)\right] e^{pz} + \left[\alpha_2 p^3 + \alpha_4 p^2 (pz - 2\mu)\right] e^{-pz}\right) p J_1(pr)\, dp$$

$$w = \frac{1+\mu}{E} \int_0^\infty \left(\left[-\alpha_1 p^2 - \alpha_3 p (pz + 4\mu - 2)\right] e^{pz} + \left[-\alpha_2 p^2 - \alpha_4 p (pz - 4\mu + 2)\right] e^{-pz}\right) p J_0(pr)\, dp$$

Boundary Conditions at the top of the surface layer are for;

$$z = 0$$

$\sigma_{zz} = -P$ for $0 \le r < a$ and $\sigma_{zz} = 0$ for $r > a$

$\sigma_{rz} = 0$ for $r \ge 0$

$$L_0(\sigma_{zz}) = -Pa \frac{J_1(pa)}{p}$$

$$L_1(\sigma_{rz}) = 0$$

$$\sigma_{zz} = \int_0^\infty \left(\left[-\alpha_1 p^3 - \alpha_3 p^2 (2\mu - 1)\right] + \left[\alpha_2 p^3 + \alpha_4 p^2 (-2\mu + 1)\right]\right) p J_0(pr)\, dp$$

$$= -Pa \int_0^\infty J_1(pa) J_0(pr)\, dp$$

$$\sigma_{rz} = \int_0^\infty \left(\left[\alpha_1 p^3 + \alpha_3 p^2 2\mu\right] + \left[\alpha_2 p^3 - \alpha_4 p^2 2\mu\right]\right) p J_1(pr)\, dp$$

$$= 0$$

Boundary Conditions at the bottom of the last layer are for;

$$z = \infty$$

$$\sigma_{zz} = 0$$

$$\sigma_{rz} = 0$$

The only non zero coefficients are; α_2 and α_4

When there is only 1 layer; α_2 and α_4 can be solved as follows;

$$\sigma_{rz} = \int_0^\infty \left(\alpha_2 p^4 - \alpha_4 p^3 2\mu\right) J_1(pr)\,dp = 0$$

$$\alpha_2 p^4 = \alpha_4 p^3 2\mu$$

$$\sigma_{zz} = \int_0^\infty \left(\alpha_4 p^3 2\mu + \alpha_4 p^3 [-2\mu + 1]\right) J_0(pr)\,dp$$

$$= -Pa \int_0^\infty J_1(pa) J_0(pr)\,dp$$

$$\alpha_4 p^3 = -PaJ_1(pa)$$

$$\alpha_2 p^4 = -PaJ_1(pa)\,2\mu$$

For $r = 0$ and $z = 0$;

$$w = \frac{1+\mu}{E} \int_0^\infty \frac{\left(PaJ_1(pa)\,2\mu + PaJ_1(pa)(-4\mu + 2)\right)}{p} dp$$

$$= \frac{2Pa\left(1-\mu^2\right)}{E} \int_0^\infty \frac{J_1(pa)}{p} dp$$

$$\int_0^\infty \frac{J_1(pa)}{p} dp = \int_0^\infty \frac{J_1(x)}{\frac{x}{a}} \frac{dx}{a} = \int_0^\infty \frac{J_1(x)}{x} dx = 1$$

For $r = 0$ and $z > 0$;

$$w = \frac{2Pa\left(1-\mu^2\right)}{E} \left[\int_0^\infty \frac{J_1(pa)\,e^{-pz}}{p} dp + \frac{z}{2(1-\mu)} \int_0^\infty J_1(pa)\,e^{-pz} dp\right]$$

Matches Harr, equation 2-5.5a; $n = z/a$

$$= \frac{2Pa\left(1-\mu^2\right)}{E} \left(\sqrt{1+n^2} - n\right) \left[1 + \frac{n}{2(1-\mu)\sqrt{1+n^2}}\right]$$

For $r > 0$ and $z = 0$;

$$w = \frac{2Pa\left(1-\mu^2\right)}{E} \int_0^\infty \frac{J_1(pa)\,J_0(pr)}{p} dp$$

Matches Harr, equation 2-5.5b;

$t = r/a, k = 2\sqrt{ar}/(a+r)$ K and E are elliptic integrals of first and second kind

$$= \frac{2Pa\left(1-\mu^2\right)}{\pi E} \left[(1+t)\,E(k) + (1-t)\,K(k)\right]$$

See Kausel, Baig, Laplace Transform of Products of Bessel Functions, 2012

5 Theoretical Results

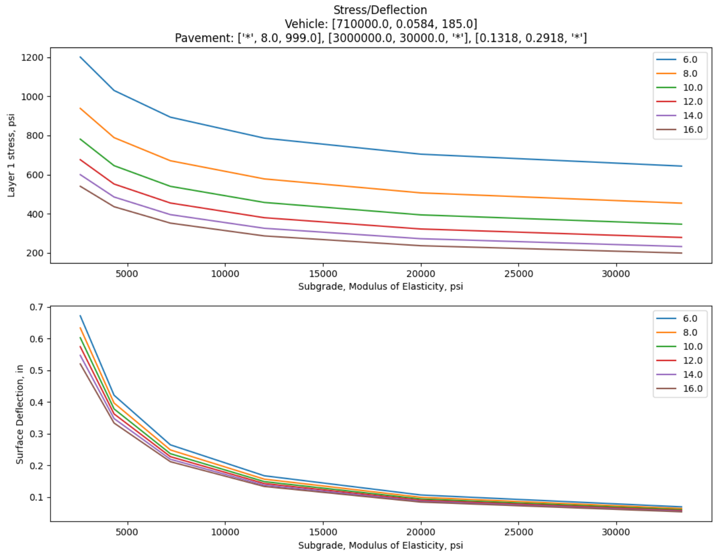

Figure 1: Stresses and Deflections

6 Opinion

An article by Jeff White, AIA, Five reasons buildings fail in an earthquake - and how to avoid them
Reason #1: The Soil Fails
Reason #2: The Foundation Fails

In my opinion, this also applies to layered systems on a soil mass.
As engineers, need to avoid weak soils if possible, remove or improve the soil.