An Identification Method for the Elastic Characterization of Materials

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ABSTRACT

A driving point dynamic load non destructive test can be used to determine a structure's first modal stiffness and indirectly the material's coefficient of elasticity. Such a dynamic test will be shown to be theoretically equivalent to a static load destructive test that directly determines the coefficient of elasticity. Assuming linear, elastic dynamic analysis is appropriate then knowing the first modal stiffness from experimental modal data and a numerical model of the structure with known geometric dimensions, an exact solution for the coefficient of elasticity can be determined when the frequency equations are available and depends on one elastic constant. The solutions for the coefficient of elasticity for uniform beams in axial deformation, torsion and bending as a function of the first modal stiffness will be given.

INTRODUCTION

There are several methods to determine the coefficient of elasticity of a pavement's sub grade soil from dynamic load non destructive tests. The frequency sweep method [8] uses the driving point frequency response to correlate to the static response of a pavement from a plate load test. The frequency sweep method uses a weighted average of the driving point frequency response to determine the equivalent static response. This paper uses the concept of correlating the response from a dynamic test to the static response from a conventional test but uses the zero frequency response of the dynamic test to compare to the static response of a conventional test. Results are shown for uniform beams but in the future will be extended to composite materials and in particular multiple elastic layered pavement systems.

DYNAMIC MODAL MODEL

The response of a multiple degree of freedom system under forced harmonic vibration ([1],[2],[6]), given by the general dynamic modal model is

$$x_s(t) = \sum_{n=1}^{N} \frac{\phi_{sn} \sum_{n=1}^{N} \phi_{dn} F_d}{K_n - \omega^2 M_n + i\omega C_n} e^{i\omega t} \tag{1}$$

For the special case when there is only one harmonic load at the sth degree of freedom and the mode shapes are normalized with respect to that location of the load, then ϕ_{sn} = 1 for n = 1,2,…,N and F_d = 0 for all d <> s and $F_d = F$ when d=s. The driving point response is

$$x(t) = H(i\omega) F e^{i\omega t} \tag{2}$$

$$H(i\omega) = \sum_{n=1}^{N} H_n(i\omega) \tag{3}$$

$$H_n(i\omega) = \frac{1}{K_n - \omega^2 M_n + i\omega C_n} \tag{4}$$

The driving point response can be written as the sum of the magnitudes and phases of the individual modes

$$x(t) = \sum_{n=1}^{N} \left| H_n(i\omega) \right| e^{i(\omega t + \theta_n)} \tag{5}$$

$$\left| H_n(i\omega) \right|^2 = \frac{1}{\left(K_n - \omega^2 M_n\right)^2 + \left(\omega C_n\right)^2} \tag{6}$$

$$\theta_n = -\tan^{-1}\left(\frac{\omega C_n}{K_n - \omega^2 M_n} \right) \tag{7}$$

The driving point response can also be written directly as the total system magnitude and phase

$$x(t) = \left| H(i\omega) \right| F e^{i(\omega t + \theta)} \tag{8}$$

$$\left| H(i\omega) \right|^2 = real\left(H(i\omega)\right)^2 + imaginary\left(H(i\omega)\right)^2 \tag{9}$$

$$\theta = \tan^{-1}\left(\frac{imaginary\left(H(i\omega)\right)}{real\left(H(i\omega)\right)} \right) \tag{10}$$

$$real\left(H(i\omega)\right) = \sum_{n=1}^{N} \frac{K_n - \omega^2 M_n}{\left(K_n - \omega^2 M_n\right)^2 + \left(\omega C_n\right)^2} \tag{11}$$

$$imaginary\left(H(i\omega)\right) = -\sum_{n=1}^{N} \frac{\omega C_n}{\left(K_n - \omega^2 M_n\right)^2 + \left(\omega C_n\right)^2} \tag{12}$$

When the load is harmonic, the response is harmonic and the magnitude and phase of the response has contributions from the individual modes. In theory, the magnitude of the frequency response at a frequency of zero corresponds to the static deflection due to a unit force. The zero frequency response [7] is defined as

$$H(0) = \left| H(0) \right| = \sum_{n=1}^{N} \frac{1}{K_n} \tag{13}$$

The zero frequency response is equivalent to the static deflection (flexibility) under a unit load of N springs in series where the spring constants, K_n, n = 1,2,…N, are the modal spring constants. The system stiffness is simply the inverse of the zero frequency response.

STATIC RESPONSE OF UNIFORM BEAM

The basic mechanical tension, compression, torsion and bending tests of uniform beams to determine the coefficient of elasticity are based on the following equations of static equilibrium [4].

Axial Tension/Compression

$$w = \frac{FL}{EA} \tag{14}$$

Torsion

$$\vartheta = \frac{TL}{GJ} \tag{15}$$

3 Point Bending at mid span

$$w = \frac{FL^3}{48EI} \tag{16}$$

Poisson's Ratio can be determined by the relation

$$\nu = \frac{E}{2G} - 1 \tag{17}$$

DYNAMIC RESPONSE OF UNIFORM BEAM IN BENDING

From the available frequency equation of a simply supported uniform beam bending at mid span ([1],[2],[6])

$$\omega_n^2 = \frac{K_n}{M_n} = \frac{n^4 \pi^4 EI}{L^4 m} \tag{18}$$

$$M_n = \frac{mL}{2} \tag{19}$$

$$K_n = \frac{n^4 \pi^4 \; EI}{2L^3} \tag{20}$$

$$\phi_n = \sin\left(\frac{n\pi x}{L}\right) \tag{21}$$

$$H(0) = \sum_{n=1}^{\infty} \frac{\phi_n^2}{K_n} \tag{22}$$

If the static response for a simply supported uniform beam bending under a static force at mid span is equal to the zero frequency response for a simply supported uniform beam bending under a dynamic force at mid span then

$$\frac{L^3}{48EI} = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)^2}{\frac{n^4 \pi^4 EI}{2L^3}} \tag{23}$$

If the above equation is to be valid then must show that

$\frac{\pi^4}{96} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ n is odd since $\sin\left(\frac{n\pi}{2}\right)^2$ is 0 when n is even and 1 when n is odd.

With the aid of Fourier series, it can be shown that the above is true.

Let $f(x) = x^4$ and expressed as an infinite series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \tag{24}$$

$$\pi a_n = \int_{-\pi}^{\pi} f(x) \cos(nx) dx \tag{25}$$

$$\pi b_n = \int_{-\pi}^{\pi} f(x) \sin(nx) dx \tag{26}$$

$$\pi a_0 = \int_{-\pi}^{\pi} f(x) dx \tag{27}$$

$$a_0 = \frac{2\pi^4}{5} \tag{28}$$

Since x^4 is an even function then none of the sine terms can be present and after integrating (25) by parts

$$a_n = \frac{48(-1)^n}{n^4} \tag{29}$$

Substituting $x = 0$ and $x = \pi$ into $f(x)$ gives two equations

$$0 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{48(-1)^n}{n^4} \tag{30}$$

$$\pi^4 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} \frac{48(-1)^n (-1)^n}{n^4} \tag{31}$$

Subtracting the (30) from the (31) has only values when n is odd

$$\pi^4 = 2\sum_{n=1}^{\infty} \frac{48}{n^4} \tag{32}$$

The static response is equal to the zero frequency response of the simply supported uniform beam in bending at mid span and can be expressed as a function of the first modal stiffness

$$\frac{L^3}{48EI} = \frac{\pi^4}{96K_1} \tag{33}$$

And finally the coefficient of elasticity as a function of the first modal stiffness is

$$E = \frac{2L^3 K_1}{\pi^4 I} \tag{34}$$

DYNAMIC RESPONSE OF UNIFORM BEAM IN AXIAL DEFORMATION

For axial deformation the available frequency equation ([1],[2],[6]) for a beam fixed at x=0 and free at x=L is

$$\omega_n^2 = \frac{K_n}{M_n} = \frac{\pi^2 (2n-1)^2 EA}{4mL^2} \tag{35}$$

$$M_n = \frac{mL}{2} \tag{36}$$

$$K_n = \frac{\pi^2 (2n-1)^2 EA}{8L} \tag{37}$$

$$\phi_n = \sin\left((2n-1)\frac{\pi}{2}\frac{x}{L}\right) \tag{38}$$

Setting the static response equal to the zero frequency response at x=L

$$\frac{L}{EA} = \frac{8L}{\pi^2 EA}\sum_{n=1}^{\infty}\frac{\sin\left((2n-1)\frac{\pi}{2}\right)^2}{(2n-1)^2} \tag{39}$$

The following can also be shown to be true by using Fourier series.

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \tag{40}$$

The static response is equal to the zero frequency response of the beam and can be expressed as a function of the first modal stiffness

$$\frac{L}{EA} = \frac{\pi^2}{8K_1} \tag{41}$$

And finally the coefficient of elasticity as a function of the first modal stiffness is

$$E = \frac{8LK_1}{\pi^2 A} \tag{42}$$

DYNAMIC RESPONSE OF UNIFORM BEAM IN TORSION

For torsion the frequency equation ([1],[2],[6]) for a cylindrical beam fixed at x=0 and free at x=L is similar to the axial frequency equation

$$\omega_n^2 = \frac{K_n}{M_n} = \frac{\pi^2 (2n-1)^2 GJ}{4mL^2} \tag{43}$$

$$G = \frac{8LK_1}{\pi^2 J} \tag{44}$$

Knowing E and G, Poisson's ratio can be determined as

$$\nu = \frac{JK_{1axial}}{2AK_{1torsion}} - 1 \tag{45}$$

DAMPING

After introducing viscous damping into the axial equations of motion [6], the coefficient of elasticity is independent of the rate of loading as determined by the dynamic test. The damping coefficient determines the response due to the rate of loading.

$$\sigma_x = E\left(\varepsilon_x + a_1 \frac{\partial \varepsilon_x}{\partial t}\right) \qquad \text{where } \varepsilon_x = \frac{\partial u}{\partial x} \tag{46}$$

$$F = EA\left(\frac{\partial u}{\partial x} + a_1 \frac{\partial^2 u}{\partial x \partial t}\right) \text{ stiffness proportional damping} \tag{47}$$

$$\frac{\partial F}{\partial x} = m\left(\frac{\partial^2 u}{\partial^2 t} - a_0 \frac{\partial u}{\partial t}\right) + q(x,t) \text{ mass proportional damping} \tag{48}$$

$$EA\frac{\partial^2 u}{\partial^2 x} + EAa_1 \frac{\partial^3 u}{\partial^2 x \partial t} - m\frac{\partial^2 u}{\partial^2 t} + ma_0 \frac{\partial u}{\partial t} = q(x,t) \tag{49}$$

Let $u(x,t) = \sum_{n=1}^{\infty} \phi_n(x) U_n(t)$ (50)

$$M_n \frac{d^2 U_n}{dt} + (a_0 M_n + a_1 K_n)\frac{dU_n}{dt} + K_n U_n = \int_0^L \phi_n(x) q(x,t) dx \tag{51}$$

So the modal damping coefficient is

$$C_n = a_0 M_n + a_1 K_n \tag{52}$$

a_0 and a_1 should be constant for all modes if the above model is valid. a_1 is a property of the material and a_0 is a property of the system.

For hysteretic damping, the coefficient of elasticity is also independent of the rate of loading. Once again the damping coefficient determines the response due to the rate of loading.

$$\sigma_x = E\left(\varepsilon_x + \frac{a_3}{\omega}\frac{\partial \varepsilon_x}{\partial t}\right) \text{ where } \omega \text{ is the excitation frequency} \quad (53)$$

SQUARE PLATE IN BENDING

A simply supported uniform plate bending due to a center load [5]

$$K_{ij} = \frac{D\pi^4}{4L^2}\left(i^2 + j^2\right)^2 \quad (54)$$

$$M_{ij} = \frac{\rho H L^2}{4} \quad (55)$$

At the center of the plate

$$D = \frac{EH^3}{12\left(1-\nu^2\right)} = \frac{4L^2 K_{11}}{\pi^4} \quad (56)$$

UNIFORM BEAM IN BENDING QUARTER SPAN

Previously the zero frequency response was not determined by the mode shapes being normalized with respect to the load point. For the uniform beam when the mode shapes are normalized with respect to the load point the following are the expected measured K_n for the experimental modal data for mid span and quarter span.

The general mode shape is

$$\phi_{sn} = \sin\left(\frac{n\pi x_s}{L}\right) \quad (57)$$

Normalized when the load is at $x_d = \frac{L}{2}$

Normalized $\phi_{sn} = \dfrac{\sin\left(\frac{n\pi x_s}{L}\right)}{\sin\left(\frac{n\pi}{2}\right)}$ which is 1 when $x_s = \frac{L}{2}$ (58)

The zero frequency response when $x_d = x_s = \frac{L}{2}$

$$\sum_{n=1}^{\infty} \frac{1}{\left(\frac{K_n}{\sin\left(\frac{n\pi}{2}\right)^2} \right)} \tag{59}$$

So measured $K_n = \dfrac{K_n}{\sin\left(\frac{n\pi}{2}\right)^2}$ (60)

When $x_d = \dfrac{L}{2}$ and $x_s = \dfrac{L}{4}$ then the zero frequency response is

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\sin\left(\frac{n\pi}{4}\right)}{\sin\left(\frac{n\pi}{2}\right)} \right)}{\left(\frac{K_n}{\sin\left(\frac{n\pi}{2}\right)^2} \right)} \tag{61}$$

and includes modes where $\sin\left(\frac{n\pi}{4}\right) <> 0$

When $x_d = x_s = \dfrac{L}{4}$ then the zero frequency response is

$$\sum_{n=1}^{\infty} \frac{1}{\left(\frac{K_n}{\sin\left(\frac{n\pi}{4}\right)^2} \right)} \tag{62}$$

So measured $K_n = \dfrac{K_n}{\sin\left(\frac{n\pi}{4}\right)^2}$ (63)

$$K_n\left(x_d = x_s = \frac{L}{4}\right) = \frac{K_n\left(x_d = x_s = \frac{L}{2}\right)}{\sin\left(\frac{n\pi}{4}\right)^2} \tag{64}$$

From statics [4], the quarter span elastic deformation is related to the mid span elastic deformation

$$w\left(x_d = x_s = \frac{L}{4}\right) = \frac{9}{16} w\left(x_d = x_s = \frac{L}{2}\right) \tag{65}$$

So

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{4}\right)^2}{n^4} = \frac{\pi^4}{96}\frac{9}{16} \tag{66}$$

Reciprocity is valid because the zero frequency response when $x_d = \frac{L}{4}$ and $x_s = \frac{L}{2}$ is equal to the zero frequency response when $x_d = \frac{L}{2}$ and $x_s = \frac{L}{4}$

The zero frequency response is

$$\sum_{n=1}^{\infty} \frac{\left(\frac{\sin\left(\frac{n\pi}{2}\right)}{\sin\left(\frac{n\pi}{4}\right)}\right)}{\left(\frac{K_n}{\sin\left(\frac{n\pi}{4}\right)^2}\right)} = \sum_{n=1}^{\infty} \frac{\left(\frac{\sin\left(\frac{n\pi}{4}\right)}{\sin\left(\frac{n\pi}{2}\right)}\right)}{\left(\frac{K_n}{\sin\left(\frac{n\pi}{2}\right)^2}\right)} \tag{67}$$

From statics [4], the quarter span deformation due to the load at the mid span is related to the mid span elastic deformation.

$$w\left(x_d = \frac{L}{2}, x_s = \frac{L}{4}\right) = \frac{11}{16} w\left(x_d = x_s = \frac{L}{2}\right) \tag{68}$$

So

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{4}\right)}{n^4} = \frac{\pi^4}{96}\frac{11}{16} \tag{69}$$

SUMMARY

The solutions for the coefficient of elasticity for uniform beams in axial deformation, torsion and bending as a function of the first modal stiffness were presented. This enables the determination of the elastic properties of uniform beams from knowledge of the first modal stiffness. The first modal stiffness is assumed to be derived from parameter estimation using experimental modal data [3]. The experimental modal data can be from any dynamic test using a time domain or frequency domain parameter estimation technique. After introducing viscous damping into the equations of motion, the coefficient of elasticity remains constant but the dynamic response is dependent on two additional constants, a stiffness proportional constant and a mass proportional constant. The stiffness proportional constant is dependent on the material and the mass proportional constant is dependent on the system.

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